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Structures with extreme relativistic cores

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Abstract. Structures with isothermal cores have been discussed in great detail in the literature, but for all these structures the value of $dP/d\rho$ jumps at the core–envelope boundary. In this paper the authors have chosen cores with extreme relativistic conditions ($dP/d\rho = 1$ and $\frac{1}{3}$). For such cores one can ensure the continuity of $dP/d\rho$ along with that of pressure, density, e^{λ} and e^{ν}. Choosing polytropic envelopes, which have positive distribution function for all possible energies provided that $dP/d\rho \leq 1$, the central redshifts have been calculated. One can obtain any high value of central redshift. The structures are pulsationally stable for $Z_c \leq 1{\cdot}43$ when the polytropic index $n = 1$. For $n < 1$, one may obtain a maximum central redshift of 4·75 for pulsationally stable structures. Next, an envelope in which the density is a specific function of r is chosen. By assuming the surface density to be equal to 2×10^{14} g cm^{-3}, the mass of neutron stars has been calculated. The maximum mass of $4{\cdot}7M_{\odot}$ is consistent with the results of other authors.

1. Introduction

Bondi (1964) discussed structures with isothermal cores and envelopes having (i) constant density and (ii) an equation of state $dP/d\rho = 1$. Das and Narlikar (1975) extended the work of Bondi (1964) by taking different isothermal cores and envelopes with equation of state $d\rho = n\, dP$. Durgapal *et al* (1980a,b) used models with isothermal core and varying density to obtain a maximum value for the central redshift in pulsationally stable structures. With his model Bondi (1964) obtained a surface redshift, $Z_s = 0{\cdot}62$.

In all the models mentioned above the continuity of pressure P, density ρ, e^{ν} and e^{λ} is assured. However, the value of $dP/d\rho$ jumps up at the boundary of core and envelope. This seems unrealistic and models become artificial. In the present paper the authors have developed a model with an extreme relativistic core (the equation of state is given by $dP/d\rho = 1$ and $\frac{1}{3}$ respectively), such that at the core–envelope boundary the continuity of P, ρ, e^{ν}, e^{λ} and $dP/d\rho$ is assured.

One may say that, for a cluster structure, continuity of $dP/d\rho$ is not necessary and the models obtained with an isothermal core are equally valid. But a cluster structure is possible only when the model considered has a positive distribution function. All the models mentioned above have a negative distribution function for quite a large range of energy (Das 1976, Durgapal *et al* 1979a,b,c). In this paper we have obtained the central redshift only for those structures which have a positive distribution function for all values of energy. Fackerell (1968) has given a method for obtaining the distribution function F for a spherical configuration with isotropic pressure, and has given internal solutions (that is, known values of ρ, P, e^{ν} and e^{λ}). Further, he has shown that polytropic gas spheres have positive distribution function when the velocity of sound

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$v_s = (dP/d\rho)^{1/2} \leqslant 1$. For this reason we have chosen the polytropic equation of state in the envelope such that $dP/d\rho \leqslant 1$.

As for compact objects, the value of $dP/d\rho$ must be continuous at the boundary and also $dP/d\rho \leqslant 1$. We have not chosen an envelope with constant density because, for $\rho =$ constant, we have $dP/d\rho = -\infty$. Hence, we have assumed a specific density distribution in the envelope given by $\rho = \rho_0(1 - r^N/K_2^N)$, and only those solutions are considered in which $dP/d\rho \leqslant 1$ and the density is non-negative.

2. Field equations and their solutions

The general assumptions made for solving the Einstein field equations are the same as those given by Bondi (1964). For a spherically symmetric and static metric,

$$ds^2 = e^\nu \, dt^2 - e^\lambda \, dr^2 - r^2 \, d\theta^2 - r^2 \sin^2\theta \, d\phi^2. \tag{1}$$

Here ν and λ are functions of r alone. The resulting field equations are

$$8\pi P = e^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2, \tag{2}$$

$$8\pi P = e^{-\lambda}[\nu''/2 + \nu'^2/4 - \lambda'\nu'/4 + (\nu' - \lambda')/2r], \tag{3}$$

$$8\pi\rho = e^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2, \tag{4}$$

where primes denote differentiation with respect to r. Equations (2)–(4) can be simplified to give

$$P' = -\nu'(P+\rho)/2, \tag{5}$$

$$P' = -(P+\rho)(m + 4\pi P r^3)/[r(r-2m)], \tag{6}$$

where

$$m = \int_0^r 4\pi\rho r^2 \, dr. \tag{7}$$

Equation (6) is the well known LOV equation for hydrostatic equilibrium.

2.1. *Core solutions: equation of state $dP/d\rho = K$*

For the equation of state $dP/d\rho = K$, we have

$$P = K(\rho - \alpha) \tag{8}$$

where α is the integration constant. For the core solutions we have chosen $K = 1$ and $\frac{1}{3}$ respectively. The coupled equations (6)–(8) are solved numerically for different values of α. We have obtained values of P, ρ, e^λ, m and dP/dr at each value of r for a constant interval Δr.

To obtain solutions in the envelope, we first determine the boundary conditions at the core–envelope boundary $r = r_i$ such that each of the parameters P, ρ, e^λ, m, dP/dr and $dP/d\rho$ is continuous at the boundary. For this we need the equation of state in the envelope. After obtaining the respective values of these parameters at the boundary (that is, values of P_i, ρ_i, e^{λ_i}, m_i, P'_i) and taking them to be the initial values, the coupled equations (6), (7) and the equation of state in the envelope can be solved numerically till the pressure vanishes. The value of $r = a$ where $P = 0$ is the radius of the entire configuration.

2.2. *A polytropic envelope*

The equation of a polytrope with polytropic index n is given by

$$P = K_1 \rho^{1+1/n} \tag{9}$$

where K_1 is a constant. Equation (9) gives

$$\mathrm{d}P/\mathrm{d}\rho = [(n+1)/n]K_1\rho^{1/n} \qquad \text{and} \qquad (P/\rho) = K_1\rho^{1/n}$$

or

$$P/\rho = [n/(n+1)](\mathrm{d}P/\mathrm{d}\rho). \tag{10}$$

For continuity of $\mathrm{d}P/\mathrm{d}\rho$ and P/ρ at the core–envelope boundary we must have

$$(P_i/\rho_i) = [n/(n+1)]K \qquad \text{and} \qquad K_1 = P_i/\rho_i^{1+1/n}. \tag{11}$$

Thus, for the core corresponding to $K=1$, the polytrope with index $n=1$ will be matched for that value of $r=b$ at which $(P_i/\rho_i)=\frac{1}{2}$ and so on. Once the value of $r=b$ is known, we take for the initial values P_i, ρ_i, m_i, P'_i and e^{λ_i} and solve the coupled equations (6), (7) and (9) till pressure vanishes at $r=a$. The surface redshift Z_s and central redshift Z_c for different polytropic indices n have been obtained.

2.3. *Envelope with density* $\rho = \rho_0(1 - r^N/K_2^N)$

In the equation

$$\rho = \rho_0(1 - r^N/K_2^N) \tag{12}$$

ρ_0, N and K_2 are constants. When we choose a particular value of N, the constants ρ_0 and K_2 can be evaluated from the continuity of ρ and $\mathrm{d}P/\mathrm{d}\rho$. In this case we have chosen the core–envelope boundary in accordance with Bondi (1964) by taking

$$H = 2v - (u^2 + v^2 + 6uv) = 0 \tag{13}$$

at $r=b$ (here $v = 4\pi P r^2$, $u = m/r$). Figure 1 shows $(u$–$v)$ tracks for different values of α appearing in equation (8). The $(u$–$v)$ tracks beyond $H=0$ are also shown for the envelope in which density varies as $\rho = \rho_0(1 - r^N/K_2^N)$. At $r = r_i$, we know the values of P_i, ρ_i, m_i, P'_i and $\mathrm{d}P/\mathrm{d}\rho = 1$ (we have chosen the core with $\mathrm{d}P/\mathrm{d}\rho = 1$ only). The continuity of ρ_i, P'_i and $(\mathrm{d}P/\mathrm{d}\rho)_i = 1$ gives

$$\rho_0 = \rho_i - P'_i r_i/N, \qquad K_2 = (-N r_i^{N-1}\rho_0/P'_i)^{1/N}. \tag{14}$$

Once the values of constants ρ_0 and K_2 are known, the coupled equations (5), (6), (7) and (12) can be solved numerically by taking ρ_i, P_i, ν_i, m_i and r_i as initial values till the pressure vanishes at $r=a$. Out of all the values of N for which solutions have been obtained, only those values have been chosen which correspond to $\mathrm{d}P/\mathrm{d}\rho \leq 1$ and non-negative density (that is, $a^N/K^N < 1$). It is noticed that the surface redshift is maximum for smaller values of N (for each value of α). If we assume ρ_s (the density at $r=a) = 2\times10^{14}\ \mathrm{g\ cm^{-3}}$ (Durgapal *et al* 1979a) we obtain the mass of neutron stars. Figure 2 shows the variation of $m/M_\odot$ with N and α. The maximum mass so obtained is $4{\cdot}7M_\odot$, which is consistent with the results obtained by Brecher and Caporasso (1976). The variation of surface redshift with N and α has also been studied (figure 3). The maximum surface redshift comes out to be $0{\cdot}86$ for $\alpha = 0{\cdot}3$.

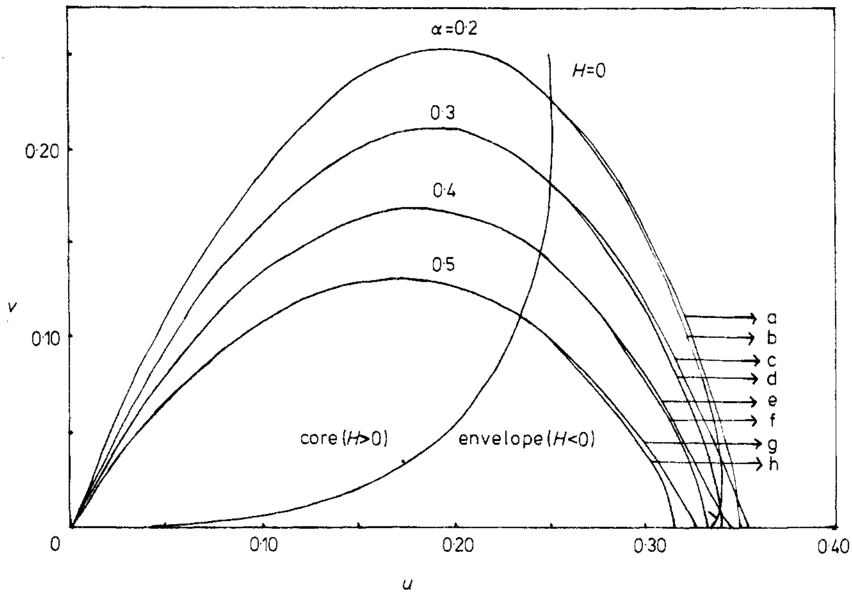

Figure 1. (u–v) tracks for envelope with varying density. a, $N \to 0$; b, $N = 0{\cdot}6$; c, $N \to 0$; d, $N = 2{\cdot}30$; e, $N \to 0$; f, $N = 4{\cdot}5$; g, $N = 0{\cdot}5$; h, $N = 7{\cdot}0$.

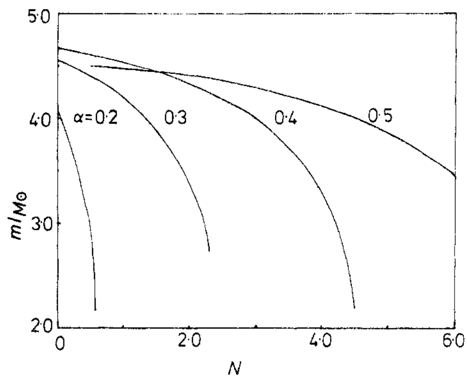

Figure 2. Variation of mass with α and N.

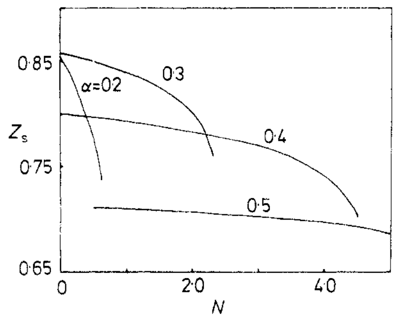

Figure 3. Variation of Z_s with α and N.

3. Distribution function

The distribution function, F, for a system with isotropic pressure and known internal solutions (that is, known values of ρ, P, e^{ν}, e^{λ}) is given by (Fackerell 1968)

$$F(x) = -(4/3\pi)\int_x^{\beta} G''(b)(b-x)^{-1/2}\,\mathrm{d}b + (1/\pi)G(\beta)(\beta-x)^{-5/2}H(\beta-x)$$
$$+(2/3\pi)G'(\beta)(\beta-x)^{-3/2}H(\beta-x)+(4/3\pi)G''(\beta)(\beta-x)^{-1/2} \qquad (15)$$

where

$$G(b)=P(b)b^2 \qquad \text{and} \qquad b=e^{\nu(r)}. \qquad (16)$$

Here primes denote differentiation with respect to b and $H(\beta - x)$ is the Heaviside unit function, equal to 1 for $x \leqslant \beta$. For polytropes, Fackerell (1968) has shown that the value of $F(x)$ is positive if $\mathrm{d}P/\mathrm{d}\rho \leqslant 1$.

For core solutions ($\mathrm{d}P/\mathrm{d}\rho = K$; $K = 1$ and $\frac{1}{3}$), if $b = \beta_c$ at the core–envelope boundary, the distribution function is given by (Durgapal *et al* 1979b):

(i) *for* $K = 1$

$$F(x) = [\rho_s \beta_c^2 / 6\pi(\beta - x)^{5/2}][5R - 2R\psi - (15 - 20\psi + 8\psi^2)] \tag{17}$$

(ii) *for* $K = \frac{1}{3}$

$$F(x) = [\rho_s \beta_c^2 / 12\pi(\beta - x)^{5/2}][3R^2 - (15 - 20\psi + 8\psi^2)] \tag{18}$$

where $R = \beta/\beta_c$ (the value of b at $r = a$ for the envelope) and $\psi = b/\beta_c$ for the core. The value of $F(x)$ is positive in the core for all the cases discussed in this paper.

4. Continuity of e^{ν} and central redshift

Equation (8) gives

$$P' = K\rho'. \tag{19}$$

From equations (5), (8) and (19) we obtain

$$e^{-\nu/2} = (1 + Z) = A \times (P + \rho)^{K/(K+1)} \tag{20}$$

where the value of the constant A is determined from the solution in the envelope.

4.1. *For polytropic envelope*

Using equations (5) and (9) it can easily be shown that

$$\begin{aligned} e^{-\nu/2} &= (1 + P/\rho)^{n+1} e^{\lambda a/2} \\ &= (1 + P/\rho)^{n+1}(1 + Z_s) \end{aligned} \tag{21}$$

where

$$1 + Z_s = e^{\lambda a/2} = e^{-\nu a/2} = (1 - 2M/a)^{-1/2}, \tag{22}$$

and M = mass of the entire configuration.

From continuity of e^{ν} at the core–envelope boundary and equations (20) and (21), we obtain

$$A = (1 + P_i/\rho_i)^{n+1}(P_i + \rho_i)^{-K/(K+1)}(1 + Z_s), \tag{23}$$

and hence the central redshift Z_c is given by

$$\begin{aligned} 1 + Z_c &= (1 + P_i/\rho_i)^{n+1}[(P_c + \rho_c)/(P_i + \rho_i)]^{K/(K+1)}(1 + Z_s) \\ &= [(2n+1)/(n+1)]^{n+1}[(P_c + \rho_c)/(P_i + \rho_i)]^{1/2}(1 + Z_s) \qquad \text{for } K = 1 \\ &= [(4n+3)/(3n+3)]^{n+1}\text{–}[(P_c + \rho_c)/(P_i + \rho_i)]^{1/4}(1 + Z_s) \qquad \text{for } K = \tfrac{1}{3}. \end{aligned} \tag{24}$$

4.2. *For envelope with density* $\rho = \rho_0(1 - r^N/K_2^N)$

The value of ν in the core is chosen as given in equation (20). Equation (5) for ν is solved numerically along with equations (6), (7) and (12) till the pressure vanishes at $r = a$. For $r = a$ we equate $e^{\nu a}$ with $(1-2M/a)$ and determine the constant A appearing in equation (20). Then the redshift at any point can be evaluated.

5. Stability under radial perturbations

Not all the results obtained here may be equally important because some of the configurations are unstable under radial perturbations. The pulsational stability of the models has been obtained by using the method given by Chandrasekhar (1964) and Harrison *et al* (1965). A spherical configuration is pulsationally stable if the integral Ω is positive, that is

$$\Omega = \int_0^a \{e^{(\lambda+3\nu)/2}[9(P+\rho)(dP/d\rho)+4r(dP/dr)-r^2(dP/dr)^2/(P+\rho)] + 8\pi\, e^{3(\lambda+\nu)/2}P(P+\rho)r^2\}r^2\, dr \geqslant 0. \tag{25}$$

The square of the pulsational frequency is obtained by dividing the integral by the following integral T:

$$T = \int_0^a e^{(3\lambda+\nu)/2}(P+\rho)r^4\, dr. \tag{26}$$

For the core solution $dP/d\rho = 1$, figure 4 shows the variation of ω^2/ρ_c with Z_c for different values of the polytropic index n. Figure 5 shows the variation of Z_c with ω^2/ρ_c for the core equation $dP/d\rho = \frac{1}{3}$ and $n = 1$.

Configurations with envelopes with density $\rho = \rho_0(1 - r^N/K_2^N)$ are found to be stable under radial perturbations.

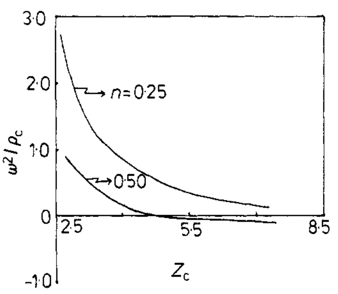

Figure 4. Variation of ω^2/ρ_c with Z_c for different polytropic indices for core equation $dP/d\rho = 1$.

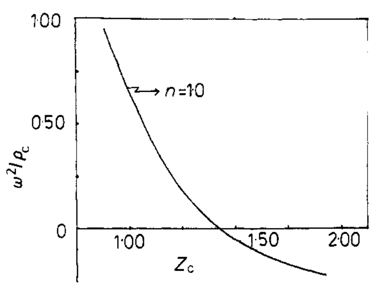

Figure 5. Variation of ω^2/ρ_c with Z_c for $n = 1$ for core equation $dP/d\rho = \frac{1}{3}$.

6. Result and discussion

(i) The solutions obtained in this paper are the most realistic two-density solutions because of the continuity of $dP/d\rho$ along with pressure, density, e^ν and e^λ. The method developed here can be considered as an extension of the application of Bondi's method (1964).

(ii) Taking $\rho_s = 2\times10^{14}$ g cm^{-3} (Durgapal *et al* 1979a), we have obtained a maximum mass of $4\cdot7M_\odot$. The results are consistent with those obtained by Brecher and Caporasso (1976) who had assumed $dP/d\rho = 1$ for all values of $\rho_s \geqslant 2\times10^{14}$ g cm^{-3}. In the present paper $dP/d\rho = 1$ for $r \leqslant r_i$, but for $r > r_i$ the value of $dP/d\rho$ decreases along with the density.

(iii) These massive configurations are found to be pulsationally stable and they correspond to a maximum surface redshift of $Z_s = 0\cdot86$. This value is higher than that given by Bondi (1964). Though the method used here is more restrictive than that used by Bondi, yet we are getting a higher value of surface redshift. This is because of the fact that in Bondi's model $P \leqslant \frac{1}{3}\rho$ for the entire configuration. However, Bondi's model with a core equation, $P = 0\cdot5\rho$, leads to a surface redshift of 0·75 (Das and Narlikar 1975) and a structure with equation of state $dP/d\rho = 1$ leads to a surface redshift of 0·91 (Durgapal *et al* 1979b).

(iv) For polytropic envelopes, the central redshift Z_c increases with α and n (figure 6). However, for higher n values the models are unstable.

(v) For extreme relativistic cores ($dP/d\rho = 1$) the structures with polytropic index $n \geqslant 0\cdot5$ are unstable. For $n \leqslant 0\cdot5$ the stable configurations are possible for $Z_c \leqslant 4\cdot75$.

(vi) For cores with $dP/d\rho = \frac{1}{3}$ we obtain stable structures with polytropic index $n \leqslant 1\cdot0$. For $n = 1$, the structures are stable for $Z_c \leqslant 1\cdot43$.

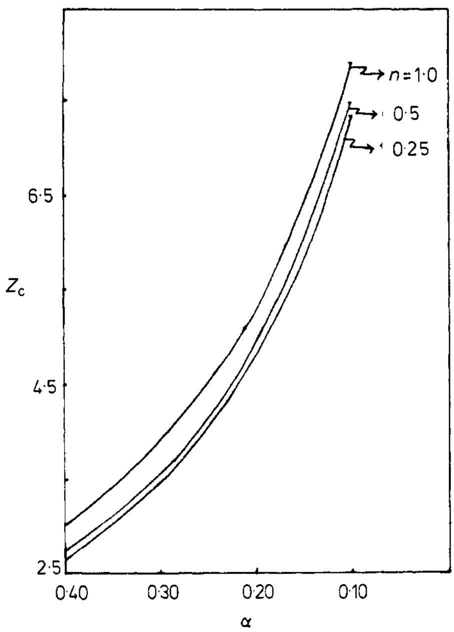

Figure 6. Variation of Z_c with α and n.

Acknowledgments

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